Open University Winter Combinatorics Meeting

Wednesday 24 January 2001

Schedule

	Welcome and introduction Fred Holroyd, Head of Pure Mathematics
10:15 - 10:55	Alex Rosa, McMaster University Specialized colourings of Steiner systems
10:55 - 11:20	Morning Tea/Coffee
11:20 - 12:00	Ian Wanless, Oxford University Cycles in Latin Squares
12:05 - 12:45	Lowell Beineke, Purdue University (visiting Oxford) Graph Decompositions with Design Connections
12:45 - 14:15	Lunch
14:15 - 14:55	Donald A. Preece, Queen Mary and Westfield College, and University of Kent at Canterbury Balanced Graeco-Latin row-column designs - an update
14:55 - 15:20	Afternoon Tea/Coffee
15:20 - 16:00	Jean Doyen, Université Libre de Bruxelles Homogeneous and ultrahomogeneous combinatorial structures

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Abstracts

Specialized colourings of Steiner systems

Alex Rosa, McMaster University, Hamilton, Ontario, Canada

Motivated by the recently introduced notion of the upper chromatic number, we consider colourings of Steiner triple systems and of Steiner systems $S(2,4,v)$ in which blocks have prescribed colour patterns, as a refinement of classical weak colourings. The main question studied is, given an integer k, does there exist a colouring of given type using exactly k colours? For several types of colourings a complete answer to this question is given while for others only partial results are obtained. A connection to maximal arcs is exhibited. We also discuss the existence of uncolourable systems.

Cycles in Latin Squares

Ian Wanless, Oxford University

A $k \times n$ *Latin rectangle* (with $k \leq n$) is a $k \times n$ matrix containing n symbols each of which occurs exactly once in each row and at most once in each column. A *Latin square* is just an $n \times n$ Latin rectangle, where n is called the *order* of the square. Another way to think of a Latin square is as a set of (row, column, symbol) triples. The Latin property insists that distinct triples never agree in more than one co-ordinate. Each Latin square has 6 *conjugate* Latin squares obtained by uniformly applying a permutation to its triples. For example, the $(2,1,3)$-conjugate is the usual matrix transpose, whereas the $(1,3,2)$-conjugate is obtained by considering each row as a permutation and replacing it by its inverse permutation.

A *row cycle* of a Latin square L is a minimal non-trivial Latin subrectangle of L. In other words it is a $2 \times m$ Latin subrectangle that contains no $2 \times m'$ Latin subrectangle for $2 \leq m' < m$. Another way to think of row cycles is this: for any two rows r_1 and r_2 of L there is a permutation of the symbol set which maps r_1 to r_2. This permutation can be written in terms of disjoint cycles in the standard way. Let c be any one of these cycles. If we look at where the symbols in c occur in rows r_1 and r_2, this is exactly a row cycle.

As well as row cycles, there are also column cycles and symbol cycles. Conjugacy interchanges these objects, which we simply call *cycles*.

There is a connection with graphical cycles too. Each Latin square of order n encodes a 1-factorisation (that is, a decomposition into 1-factors) of $K_{n,n}$. Cycles in the Latin square correspond to the cycles you get when you take the union of two 1-factors in the 1-factorisation. A 1-factorisation is said to be *perfect* if the union of any two 1-factors is a Hamiltonian cycle. A Latin square which encodes a perfect 1-factorisation of $K_{n,n}$ is said to be *pan-Hamiltonian*. Such squares are N_∞, meaning they never have non-trivial subsquares.

Pan-Hamiltonian Latin squares have just one type of cycle (say the row-cycles) which are Hamiltonian, but it is also possible to find Latin squares in which all cycles are Hamiltonian. Such squares are called *atomic* Latin squares. The most obvious examples are the Cayley tables of cyclic groups of prime order, but other infinite families are known.

In this talk I will discuss recent results on pan-Hamiltonian and atomic Latin squares, plus perfect 1-factorisations of complete bipartite graphs. I will focus on the existence questions, but also mention a few 'applications'. Much of what I discuss will be the fruits of collaboration with Barbara Maenhaut (Open University) and Darryn Bryant (University of Queensland).

Graph Decompositions with Design Connections

Lowell Beineke, Purdue University, Indiana, USA (visiting Oxford)

If F and G are graphs, an F-decomposition of G is a partition of the edges of G into copies of F, and an F-factorisation of G is an H-decomposition in which H has the same order as G and each of its components is a copy of F.

One way to view block designs is as complete graph decompositions. Taking this as our starting point, we will proceed to discuss a variety of results on graph decompositions and factorisations. The main constituent graphs that we will consider, in addition to complete graphs, will be cycles and paths.

We will also go beyond decompositions to discuss some problems on graph packings, which can be thought of as partial decompositions. Here, we will focus our attention on complete bipartite graphs as well as complete graphs.

Balanced Graeco-Latin row-column designs - an update

Donald A. Preece, Queen Mary and Westfield College, and University of Kent at Canterbury

The card-game Five Crowns is played with 2 packs, each comprising, apart from Jokers, 11 cards (with values 3, 4, ... , 10, J, Q, K) for each of five suits (stars, hearts, clubs, spades, diamonds). The 55 cards can be laid out as follows:

5♣	9♢	3♣	10♡	K♡	8♠	7⋆	6♠	Q♢	J⋆	4⋆
3⋆	6⋆	7♢	9♠	10♣	J♣	K♢	8♡	4♠	Q♡	5♡
8♢	3♡	4♡	Q⋆	7♠	10⋆	5♠	J♢	9♣	K♣	6♣
K♠	J♠	Q♠	8♣	9⋆	7♡	6♡	4♣	5⋆	3♢	10♢
9♡	7♣	8⋆	6♢	4♢	5♢	Q♣	K⋆	J♡	10♠	3♠

This 5×11 Graeco-Latin design is a *double Youden rectangle* (DYR), characterised by its overall balance. Each value appears exactly once per row, and each suit appears exactly once per column. Each pair of values occurs within exactly λ columns, $\lambda = 2$; each suit occurs exactly n times in each row bar one, $n = 2$, and occurs $n + 1$ times in the remaining row. An analogous 4×13 DYR with $\lambda = 1$ and $n = 3$ can be formed from the 52 cards of a standard pack.

A further 5×11 arrangement of ordered pairs is the following:

1, 2	2, 3	3, 4	4, 5	5, 6	6, 7	7, 8	8, 9	9, 10	10, 0	0, 1
3, 6	4, 7	5, 8	6, 9	7, 10	8, 0	9, 1	10, 2	0, 3	1, 4	2, 5
9, 7	10, 8	0, 9	1, 10	2, 0	3, 1	4, 2	5, 3	6, 4	7, 5	8, 6
5, 10	6, 0	7, 1	8, 2	9, 3	10, 4	0, 5	1, 6	2, 7	3, 8	4, 9
4, 8	5, 9	6, 10	7, 0	8, 1	9, 2	10, 3	0, 4	1, 5	2, 6	3, 7

The elements x in the pairs (x, y) in the first column constitute a difference set mod 11, with $\lambda = 2$, as do the elements y and also the elements $y - x$ mod 11. Thus, throughout the design, which is developed cyclically from the first column, the elements x are balanced with respect to columns, as are the elements y; further, the elements x are balanced with respect to the elements y, and *vice versa*. These properties of balance, and a further overall balance property, characterise the design as a *Freeman-Youden rectangle* (FYR).

DYRs and FYRs constitute the principal classes of balanced Graeco-Latin row-column designs.

The current state of knowledge of constructions for DYRs and FYRs is reviewed. Further work on DYRs is much needed.

Homogeneous and ultrahomogeneous combinatorial structures

Jean Doyen, Université Libre de Bruxelles, Belgium

A set S provided with a certain structure is said to be **homogeneous** if, whenever the substructures induced on two finite subsets S_1 and S_2 are isomorphic, there is at least one automorphism of S mapping S_1 onto S_2. If every isomorphism from S_1 to S_2 can be extended into an automorphism of S, S is called **ultrahomogeneous**.

Historically, the notions of homogeneity and ultrahomogeneity arose from ideas of Leibniz about the structure of our universe and also from the Helmholtz-Lie principle of free mobility of rigid bodies. Georg Cantor had already pointed out in 1895 that the ordered set of rational numbers is ultrahomogeneous. Another surprising example is the famous countable random graph discovered in 1963 by Erdös and Rényi.

The talk will give a short survey of the subject, for non specialists, focussing on such combinatorial structures as graphs, partial linear spaces and t-designs.